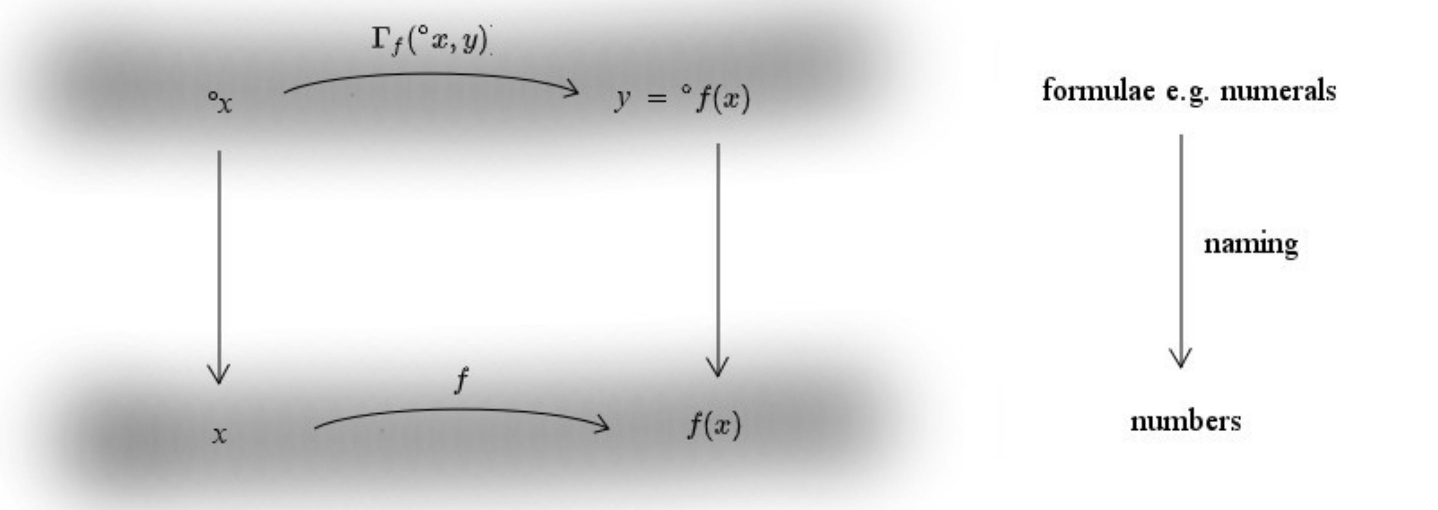
Background [edit]

Let **N** be the set of natural numbers. A theory *T* **represents** the computable function *f* : **N**→**N** if there exists a "graph" predicate $\Gamma_f(x,y)$ in the language of *T* such that for each **x** in **N**, *T* proves

$$\forall y \quad ({}^\circ f(x) = y \Leftrightarrow \Gamma_f({}^\circ x, y)).$$ [3]

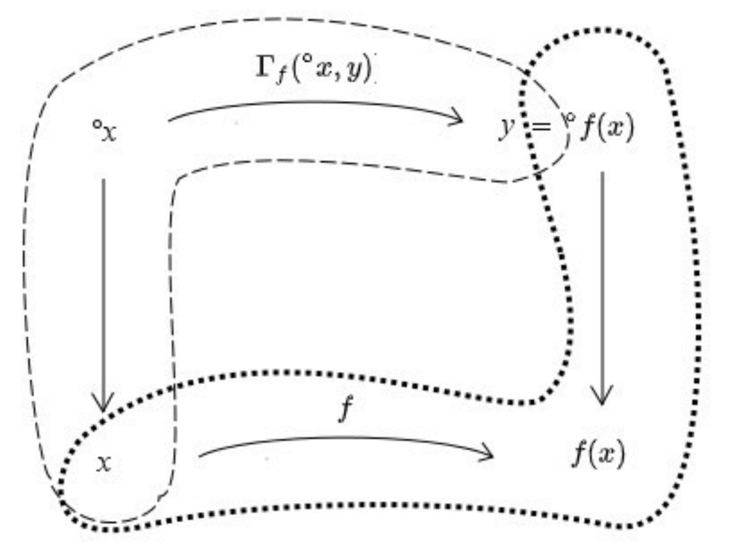
Here °**x** is the numeral corresponding to the natural number **x**, which is defined to be the closed term 1+ ··· +1 (**x** ones), and °*f*(**x**) is the numeral corresponding to *f*(**x**).



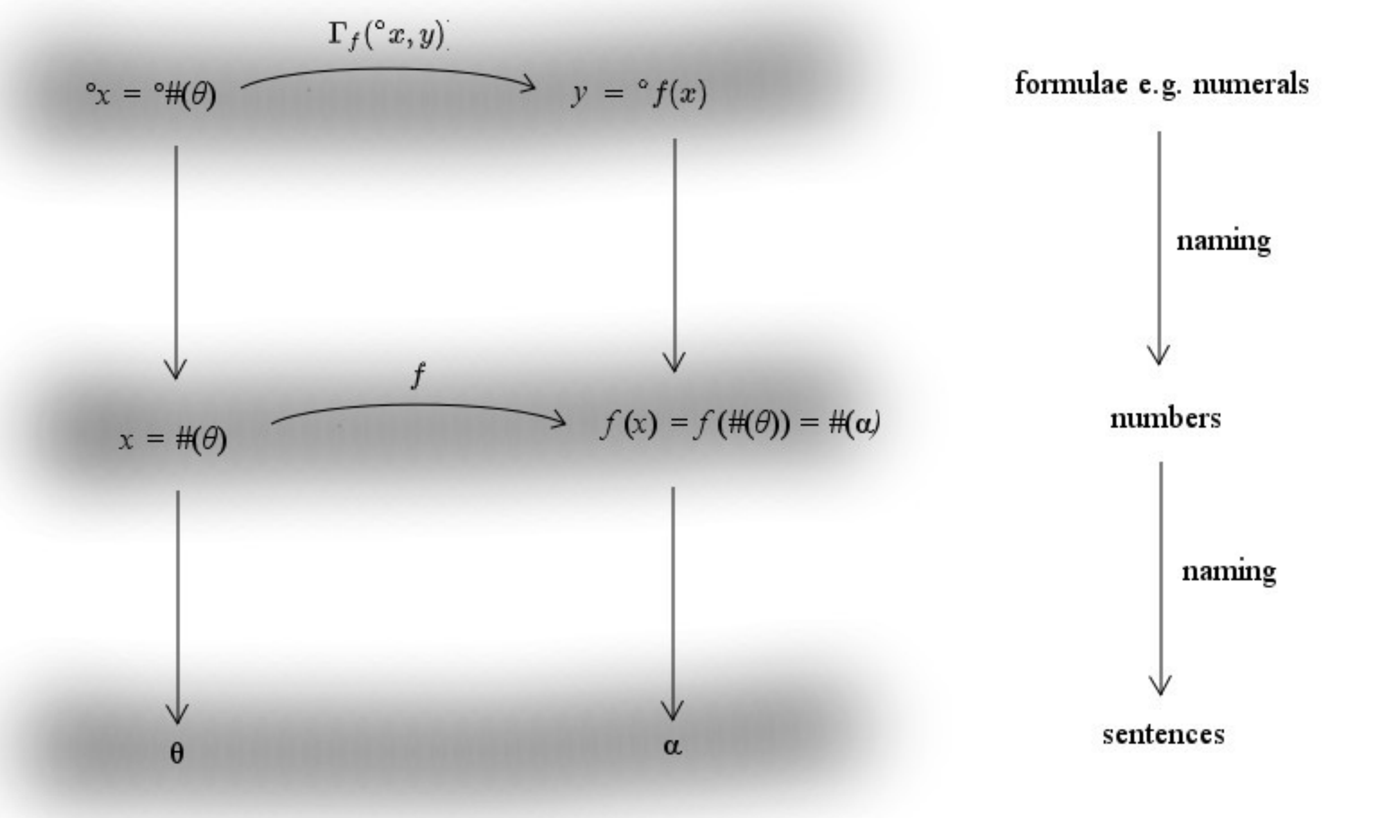
i.e. T proves,

y = (x) = (x)

or, f(x) = (x)



The diagonal lemma also requires that there be a systematic way of assigning to every formula θ a natural number #(θ) called its Gödel number. Formulas can then be represented within the theory by the numerals corresponding to their Gödel numbers. For example, θ is represented by °#(θ)



> **Lemma** — There is a sentence ψ such that ψ ↔ F(°#(ψ)) is provable in *T*.[4]

Proof [edit]

Let *f*: **N**→**N** be the function defined by:

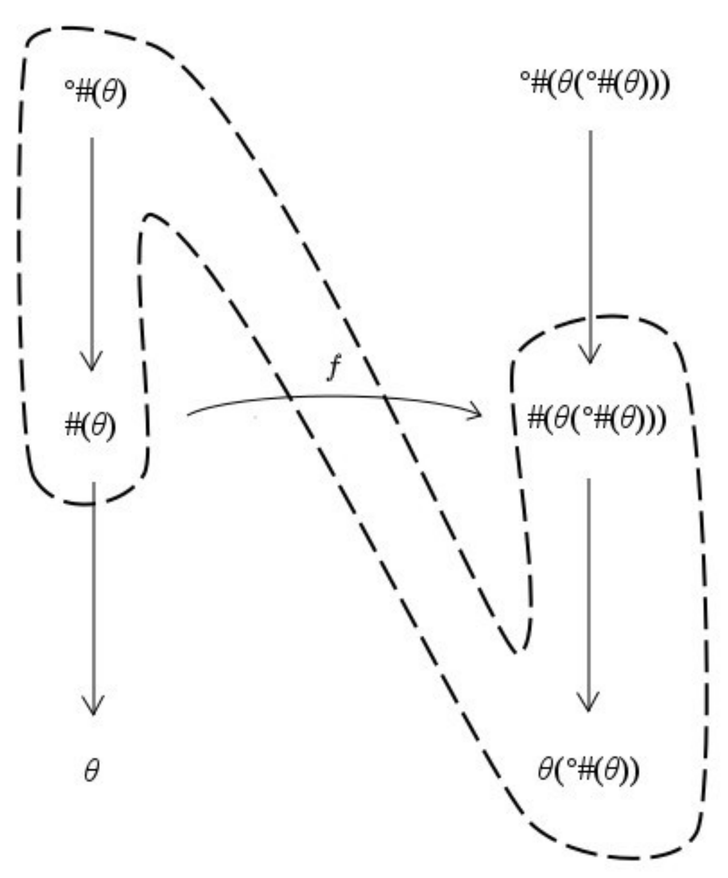
$f(\#(\theta)) = \#(\theta({}^\circ\#(\theta)))$

for each *T*-formula θ in one free variable, and *f*(*n*) = 0 otherwise.

i.e. $(\#(\theta)) = f(\#(\theta)) = (\theta) = \#(\theta({}^\circ\#(\theta)))$

Also, $(\theta) = {}^\circ f(\#(\theta)) = {}^\circ\#(\theta({}^\circ\#(\theta)))$

and $(\theta) = \theta({}^\circ\#(\theta))$



The function *f* is computable, so there is a formula Γ_f representing *f* in *T*. Thus for each formula θ, *T* proves

$(\forall y)\ [\Gamma_f({}^\circ\#(\theta),y) \leftrightarrow y = {}^\circ f(\#(\theta))]$,

which is to say

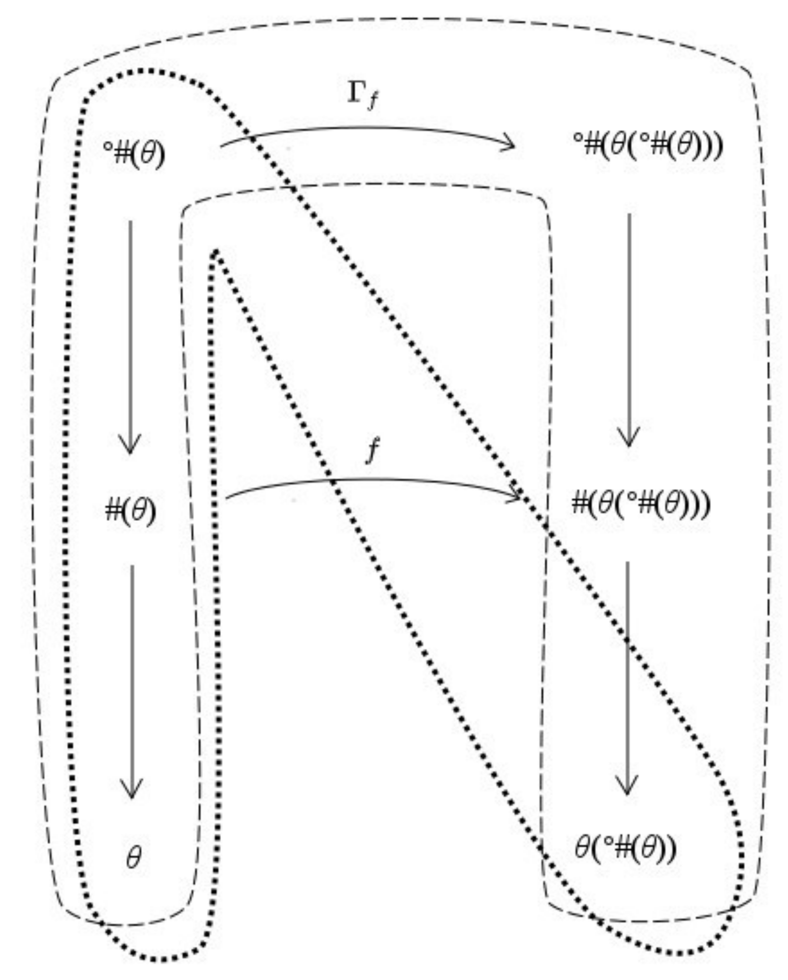
$(\forall y)\ [\Gamma_f({}^\circ\#(\theta),y) \leftrightarrow y = {}^\circ\#(\theta({}^\circ\#(\theta)))]$.

i.e. $(\theta) = (\theta) = {}^\circ\#(\theta({}^\circ\#(\theta)))$

Also, $(\theta) = (\theta) = \#(\theta({}^\circ\#(\theta)))$

and $(\theta) = (\theta) = \theta({}^\circ\#(\theta))$

i.e. the result of sending θ through the u-bend is the same as the result of sending θ through the v-bend.



Now define the formula β(*z*) as: $\beta(z) = (\forall y)\ [\Gamma_f(z,y) \rightarrow F(y)]$ = F((θ)) ... but more intuitive to use: F((θ)) i.e.: the result of sending θ through the u-bend has property F, e.g. is false

Then *T* proves

$\beta({}^\circ\#(\theta)) \leftrightarrow (\forall y)\ [\, y = {}^\circ\#(\theta({}^\circ\#(\theta))) \rightarrow F(y)]$,

which is to say

$\beta({}^\circ\#(\theta)) \leftrightarrow F({}^\circ\#(\theta({}^\circ\#(\theta))))$.

i.e. T proves: F((θ)) ⇔ F((θ))

i.e. T proves that saying: the result of sending θ through the u-bend is false

is equivalent to saying: the result of sending θ through the v-bend is false

Now take θ = β and ψ = β(°#(β)), and the previous sentence rewrites to ψ ↔ F(°#(ψ)), which is the desired result.

i.e. F[(F((θ)))] i.e. [the result of sending "the result of sending θ through the u-bend is false" through the u-bend] is false

⇔ F[(F((θ)))] i.e. [the result of sending "the result of sending θ through the u-bend is false" through the v-bend] is false

⇔ F [F ((F((θ))))] i.e. "the result of sending "the result of sending θ through the u-bend is false" through the u-bend is false" is false